

Dynamic Calculus Software: Constructing Calculus Concepts with Sketchpad™ Version 4

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One description of calculus is “the mathematics of change.” The study of limits and instantaneous rate of change, and readily available applications in kinematics and dynamics in physics, make this description an apt one for beginning calculus students.

Another feature of calculus is the visual nature of many of the topics it encompasses. Calculus texts abound with pictures and graphs showing moving objects, showing functions and their rates of change, showing areas under curves, and showing discs, shells and washers of solids of revolution.

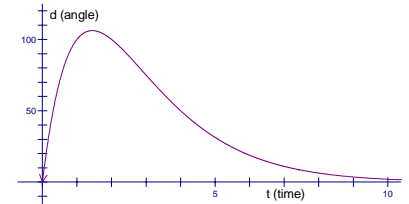
Dynamic mathematics visualization software such as The Geometer’s Sketchpad® can make the dynamic and visual aspects of calculus come alive for your students. In this session we’ll examine the elements required to make this happen successfully in your classroom, and we’ll look at several lab and demonstration activities that you can use.

This session, including the presentation sketch, the activities, and a commentary, is available electronically at <http://www.keptech.com/sketchpad/scott/nctm2004>.

Many schools have doors with automatic closers. When you push such a door, it opens quickly, and then closes more and more slowly until it closes completely.

Door Angle as a Function of Time

1. Open **Instantaneous Rate.gsp** in the **9 Introduction to Calculus** folder. Press the *Open Door* button to observe the door opening and closing. The graph shows the angle of the door in degrees (d) as a function of time (t).



2. Drag point t_1 along its axis, watching both the angle of the door and the point on the graph. Observe the values of the t_1 and d_1 measurements as you drag.
 - Q1 For what values of t_1 is the angle increasing? How can you tell?
 - Q2 What is the maximum angle the door reaches? At what time does this occur?

The Door at Two Different Times

The value Δt is the separation between the two values of time (t_1 and t_2).

3. To find the rate of change of the door's angle, you need to look at the door's position at two different times. Press the *Show t_2* button to see a second point on the time axis slightly separated from point t_1 . Drag point t_1 back and forth, and observe the behavior of the new points on the graph. To change the separation of the two times, press the button labeled *1.0*, and then the button labeled *0.1*.
 - Q3 What is the largest separation you can get by moving the slider? What's the smallest separation you can actually observe on the graph?
4. Make the separation of the two points smaller than 0.1. Can you still see two distinct points on the graph? Can you see the values of t_2 and d_2 change as you make h smaller? Experiment with dragging the Δt slider, to change the separation of the two values of time directly.
 - Q4 As you make Δt smaller, can you observe changes in the numeric values of t_2 and d_2 even when you can no longer observe any changes on the graph?

The Rate of Change of the Door's Angle

Hint: Divide the change in the angle by the change in the time.

5. Set Δt to 0.1, and then use the numeric values of t_1 , d_1 , t_2 and d_2 to calculate the rate of change of the door's angle at any particular time. (Use Sketchpad's Calculator to do this calculation.)

- Q5 What are the units of the rate of change? What does the rate of change tell you about the door's motion?
6. Press the *Show Rate* button to check your result.
- Q6 What's the relationship between the dotted line and the rate of change you calculated?
- Q7 Move t_i back and forth. How can you tell from the rate of change whether the door is opening or closing? How can you tell whether its rate is fast or slow?
- Q8 Use the buttons to set t_i to 1.0 and Δt to 0.1. What's the rate of change?
7. Select the numeric values of t_i , d_1 , t_2 , d_2 , Δt and the rate of change. With these six measurements selected, choose **Graph | Tabulate**. Double-click the table to make the current entries permanent.

When you press this button a dotted line appears connecting the two points on the graph.

The value of t_i should now be exactly 0.10000.

You should now have two rows of numbers in the table, with the first row permanent and the second row changing as the measurements themselves change.

The Limit of the Rate of Change

You may want to press the 0.1 button and then the 0.01 button again to check the motion of the dotted line.

8. Set the time interval (Δt) to exactly 0.01. Note the new value of the rate of change. Could you see the dotted line move as you reduced the time interval? With the interval set to 0.01, double-click the table to permanently record these new values.
- Q9 How does this rate of change compare to the value when Δt was 0.1?
9. Record in the table values for intervals of 0.001, 0.0001, 0.00001, and 0.000001
- Q10 What do you notice about the value of the rate of change as the time interval becomes smaller and smaller? What value does the rate of change seem to be approaching?
- Q11 Can you see the dotted line move as Δt changes from 0.001 to 0.0001?
10. Set the value of t_i to 3 seconds (by pressing the $t \rightarrow 3$ button), and collect more data on the rate of change of the door's angle. Collect one row of data for each time interval from 0.1 second to 0.000001 second.

The *average rate of change* is the rate of change between *two* different values of t . The *instantaneous rate of change* is the exact rate of change at *one* specific value of t . Since you must have two different values to calculate the rate of change, one way of measuring the instantaneous rate of change is by making the second value closer and closer to the first, and finding the *limit* of the average rate of change as the interval gets very small.

The instantaneous rate of change of a function – that is, the limit of the average rate of change as the interval gets close to zero – is called the *derivative* of the function.

- Q12 What is the derivative of the door's angle when t_i is 3 seconds?

If $g(x)$ is the derivative of $f(x)$, then $f(x)$ is the antiderivative of $g(x)$.

You've already learned about the derivative of a function, and you know that the value of the derivative is the slope of the line that's tangent to the graph of the function. In this activity, you'll reverse the process, and start with the derivative function. Your job is to find the original function for which the given function is the derivative. This original function is called the *antiderivative*. You'll do this by building a probe based on the relationship between a derivative and the slope of a tangent line.

Creating the Function

Use the Values pop-up in the Calculator to create parameters a , b , c and d .

1. In a new sketch, choose **Graph | Plot New Function** to graph the function $f(x) = a x^3 + b x^2 + c x + d$. Set these values for the parameters: $a = 0.05$, $b = 0.10$, $c = -1$, $d = -1$. This function is the derivative of the function you want to find.
- Q1 For what values of x is the value of the derivative positive? When the derivative is positive, what does this tell you about the antiderivative?

Building the Probe

The probe will be a short segment pointing in the same direction as a tangent line to the antiderivative. The slope of this tangent is equal to the value of the derivative.

- Q2 When the derivative is positive, in what direction would you expect the probe to point? What direction would you expect it to point when the derivative is zero?
2. Construct independent point P , and measure its x - and y -coordinates. This is the point at which you'll construct the probe.
 3. Calculate the value of $f(x_P)$ – the value of the function at point P .
- Q3 What's the relationship between the value of $f(x_P)$ and the slope of the line that's tangent to the unknown antiderivative function at x_P ?

To construct a line through P with the correct slope, you need to find a second point on the tangent line. The first point (point P) has coordinates (x_P, y_P) , so you can think of a second point as $(x_P + \Delta x, y_P + \Delta y)$.

- Q4 Remember that the slope of a line can be expressed as $m = \Delta y / \Delta x$. If you use 1 as the value of Δx , what would be the value of Δy ?
4. Calculate the two values $x_P + 1$ and $y + f(x_P)$, and plot point Q at $(x_P + 1, y_P + f(x_P))$. Construct a line through points P and Q .
 5. Drag point P around, and observe how the slope of the line changes depending on where point P is.

- Q5 For what values of x does the line point up to the right? For what values of x is it horizontal? For what values does it point down to the right?

Although this line points in the correct direction, it's too long to make a good probe. A probe should be a short segment starting from point P .

Now you'll use the line to construct a short probe, using a circle centered at P and with a radius of 0.5.

6. Create a parameter $h = 0.5$ to determine the length of the probe segment.
 7. Plot point R at $(x_P + h, y_P)$ to create a point exactly 0.5 away from point P .
 8. Construct a circle centered at P and passing through R , and construct the right-most intersection S of the circle with the tangent line. Then hide the tangent line, the circle, and points Q and R .
 9. Finally, construct segment PS to be your probe. Drag point P around again, and observe the behavior of the probe.
- Q6 For what values of x does the probe point down? Where is it horizontal? Explain these observations based on the behavior of the function $f(x)$.

Using the Probe

Because point P snaps to integer values, the segment's traces appear at integer values on the grid.

10. Turn on tracing for the segment, and choose **Graph | Snap Points**. Drag P all around the screen, and observe the traces left behind.

- Q7 What do the traces indicate about the function whose derivative is $f(x)$?

Now you'll use the probe to trace out a solution to the original problem.

11. Turn off **Snap Points**, drag P so its x -coordinate is approximately -7 , and erase the existing traces.
12. Drag point P so that it follows the direction of the probe. In other words, if the probe is pointing up and to the right, drag P up and to the right, trying to follow the probe segment. As you drag, the slope of the probe will change; as the slope changes, continue dragging P in the direction that the probe points.
13. Practice following the probe several times, erasing traces before each new try. See how good you can get at following the traces accurately.
14. Once you have a good smooth trace, leave it on the screen, and quickly move the probe so it's directly above or below the starting point for the existing trace. Without erasing the original trace, make another trace.
15. Make a third trace, again starting at the same x position but a different y position.

This second trace should start from the same x value as the original, but start at a different y value.

- Q8 What do you notice about the shapes of the three traces?

16. Save your finished sketch. You'll need it for the next activity.

In the last activity you created a probe and used it to trace out the antiderivative of a function. In this activity you'll automate the process of tracing the antiderivative. You'll do this in two ways, using a Movement button and using the Iterate command.

Open the sketch you saved in the last activity. This sketch contains the function $f(x)$ and a probe from point P to point S whose slope matches the value of $f(x_P)$.

Tracing With a Movement Button

Make sure that **Trace Segment** is turned on for the probe.

If point P has gone off-screen, you can get it back by choosing **Edit | Undo Translate Point**.

1. Select points P and S , in that order, and choose **Edit | Action Button | Movement**. Accept the default settings for the Movement button.
2. Drag point P to an appropriate starting x-value. Press the *Move P->S* button, and point P will automatically move in the direction of S . As the slope of the probe changes, point P continues to move toward the new position of S , so that the Movement button creates a trace automatically.
3. Once your trace has gone far enough to the right, click the button again to stop the movement.

Q1 How does this automatic trace compare to the manual traces you did previously?

4. Erase the existing traces, and create five new traces of the antiderivative, each one starting from the same x-coordinate but a different y-coordinates.

Q2 What do you notice about the shapes of the five traces?

Tracing by Iteration

The traces you made manually and with the *Move P->S* button have not been dynamic objects; once you've made a trace, you can't do anything with it except to erase it and make a new one. In the next few steps, you'll use iteration to create a permanent trace that allows you to move the starting point around.

5. Turn off tracing for the probe segment. Don't erase the existing traces just yet.
6. Position point P at the same x-value that you've been using.
7. With point P selected, choose **Transform | Iterate**. In the Iteration dialog box, designate point S as the image point by clicking on point S in the sketch.
8. Click Iterate in the dialog box to iterate the probe three times. The first probe (the pre-image) goes from P to S ; the first image probe is constructed using the same construction, but beginning at point S ; the second image starts at the ending point of the first image, and so on.

The iteration consists of iterated images of both point P and the segment connecting P and S .

9. With the iterated image selected, press the + key on your keyboard several times to increase the number of iterations.
- Q3 How does the shape of the iteration compare with the existing traces on the screen?
- The iteration may look somewhat crude compared to the traces, because each step of the iteration is a segment that's longer than the steps point P took when you pressed the action button. Fortunately, you can reduce the size of the steps in the iteration.
10. Change the value of h to 0.1.
- Q4 Describe what happens to the iteration. (You should note at least two important changes.) Explain why changing the value of h has this effect.
11. To increase the number of steps in the iteration, you can select the iteration and press the + key on the keyboard, but with this method it will take you some time to create 1000 iterations. Select the iterated image of either the point or the segment and choose **Edit | Properties**. Use the Iteration pane to change the number of iterations to 1000.
12. Change the value of h to 0.01.
- Q5 How does this change the shape of the iteration? Do you think this shape is a more accurate depiction of the antiderivative? Why?
- Q6 This improved iteration is still an approximation. What would you have to do to make the shape exactly correct, to any desired degree of accuracy?
13. Move point P vertically on the screen, so that the iteration starts at a new y -value.
- Q7 How does the shape of the antiderivative change as you move P up and down?
- Q8 For each different position of point P , there's a different antiderivative. How many solutions are there to the question "Find a function whose derivative is $f(x)$?" Explain your answer.

Explore More

1. Change the original function $f(x)$ by changing the values of the parameters a , b , c and d , and note how the shape of the antiderivative relates to the changed shape of the function itself.
2. Change the original function to $f(x) = a \sin (bx + c) + d$. What shape of the antiderivative results? Try different values of a , b , c and d to experiment with different sinusoids.

INSTANTANEOUS RATE (PAGE #)

Objective: Use this activity for several purposes:

- making a connection between instantaneous rate and slope of the tangent to the graph,
- getting students to see the instantaneous rate as a limit of the slope between two points, just as the tangent represents the limit of a secant line.
- introducing the concept and definition of the derivative

Prerequisites: Students should be familiar with finding the slope given two points on a graph.

Sketchpad Proficiency: Beginner. This activity requires no construction.

Class Time: 30–40 minutes.

Required Sketch: *Instantaneous Rate.gsp*.

Door Angle as a Function of Time

- Q1 The angle increases from $t = 0$ to approximately $t = 1.45$. You can tell because the maximum value of the angle occurs at approximately $t = 1.45$.
- Q2 The maximum angle is approximately 106° , at $t = 1.45$.

The Door at Two Different Times

- Q3 The slider has a maximum value of 1. It uses a logarithmic scale, so it's easy to achieve very small values – values that are much smaller than can be observed on the graph.
- Q4 Yes, the displayed values (to 5 decimal digits) continue to show the changes.

The Rate of Change of the Door's Angle

- Q5 The units are degrees per second. The rate of change tells you by how many degrees the door is opening or closing for every second.
- Q6 The rate of change is the slope of the dotted secant line. As the secant line approaches tangency, the calculated rate of change approaches the instantaneous rate of change.
- Q7 When the rate of change is positive, the door is opening; when the rate is negative, it's closing. When the rate of change is close to zero, the door is

moving slowly; when the absolute value of the rate of change is large, it's moving quickly.

- Q8 When $t_1 = 1$ and $\Delta t = 0.1$, the calculated rate of change is 26.33629 degrees/s.

The Limit of the Rate of Change

- Q9 The average rate of change is higher (over 30 degrees/second), and appears to be a more accurate value for $t_1 = 1.0$.
- Q10 The rate of change seems to be approaching a limit; the differences are less and less. The rate of change seems to be approaching 30.685 degrees/s.
- Q11 No, the change is too small to observe on the graph.
- Q12 The derivative (that is, the limit of the rate of change) seems to be about -26.986 degrees/s. The negative sign means that the door is closing.

MANUALLY PROBING THE ANTIDERIVATIVE

Objective: Students will use what they know about derivatives and slopes to trace out an antiderivative.

Prerequisites: Students must understand the relationship between the derivative of a function and the slopes of tangents to the function.

Sketchpad Proficiency: Intermediate. Students will be expected to be able to complete simple constructions without being told what to do at each step along the way.

Class Time: 20–30 minutes. If students finish early, they could start on the related activity (Automatically Probing the Antiderivative). Most students can finish both of these activities in a 45 or 50 minute period.

Required Sketch: None.

Creating the Function

1. Make sure students know how to create parameters when constructing a function.
- Q1 The derivative is positive between about -5 and -1 , and also for values of x greater than about 4 . When the derivative is positive, the antiderivative is increasing.

Building the Probe

- Q2 When the derivative is positive, the probe should point up and to the right. When the derivative is zero, the probe should be horizontal.
- Q3 The value of $f(x)$ will be equal to the slope of the antiderivative at the same x value.
- Q4 If you use 1 as the value of Δx , the value of Δy will be equal to the value of $f(x)$.
- Q5 The line points up to the right when x is between -5 and -1 , and also when it's greater than 4 . These are the same values from Q1.
- Q6 The probe points down when x is less than -5 , and also when it's between -1 and 4 . The probe is horizontal when $f(x) = 0$, which happens at approximately -5 , -1 and 4 .

Using the Probe

- Q7 The traces indicate the direction of a possible tangent line at each spot on the screen. Students are creating a slope field here; it's up to the teacher whether to draw special attention to this and name it at this time.
12. In this step, students are using a form of Euler's method: calculate the direction in which to move (using the probe), move a short distance in the calculated direction, and then recalculate. Though there's no need to name the method, the teacher should call attention to this technique for approximating solutions when only local information is available.
- Q8 The three traces should all seem to have roughly the same shape, though they are displaced vertically from each other.

AUTOMATICALLY PROBING THE ANTIDERIVATIVE

Objective: Students will continue the previous activity by automating the tracing process, using a Movement button and using the Iterate command.

Prerequisites: Students must have completed the activity on Manually Probing the Antiderivative, and must have saved their sketch from that activity.

Sketchpad Proficiency: Intermediate. Students will be expected to be able to complete simple constructions without being told what to do at each step along the way.

Class Time: 20–30 minutes. If students finish early, the Explore More section suggests several extensions.

Required Sketch: Students must have completed and saved the sketch for the activity Manually Probing the Antiderivative.

Tracing with a Movement Button

- Q1 The automatic trace will be smoother than the manual traces, but should have approximately the same shape.
- Q2 The five traces all have the same shape. Because of the greater accuracy of automated tracing, the fact that the shapes are identical should be even clearer than when the students did the traces manually.

Tracing by Iteration

- Q3 The iterated shape is similar to the one produced by the Movement button, but is not as smooth or accurate. This is because the iteration is using much larger steps than the Movement button used.
- Q4 When the value of h becomes smaller, the segments become shorter. This smooths the jagged nature of the original iteration, but also shortens the iteration itself, so that it shows very little of the solution.
- Q5 With shorter steps at each stage of the iteration, the result is now extremely smooth and accurate. Because the direction is evaluated so frequently, there's much less opportunity for errors to accumulate.
- Q6 The shorter the steps, the smaller the errors that distort the final shape. To get a completely

accurate shape, we'd need to find the limit of the shape as the length of the steps approaches zero (and the number of steps increases without limit).

- Q7 As you move P up and down, the shape doesn't change at all.
- Q8 There are an infinite number of solutions, because for any given x -value, each possible y -value generates a different solution. Explanations vary. In terms of the probe, the solution you get depends on the y -value at which you start the probe. Stepping back from the probe, a more general answer would be that if it's the slope that matters, then the original function can be translated up or down without affecting its derivative.